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Error estimates of Lagrange interpolation and orthonormal expansions for Freud weights

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Abstract

Let $S_n[f]$ be the nth partial sum of the orthonormal polynomials expansion with respect to a Freud weight. Then we obtain sufficient conditions for the boundedness of $S_n[f]$ and discuss the speed of the convergence of $S_n[f]$ in weighted L^p space. We also find sufficient conditions for the boundedness of the Lagrange interpolation polynomial $L_n[f]$, whose nodal points are the zeros of orthonormal polynomials with respect to a Freud weight. In particular, if $W(x) = \mathrm{e}^{-(1/2)x^2}$ is the Hermite weight function, then we obtain sufficient conditions for the inequalities to hold:

$$\|(S_n[f] - f)^{(k)} W u_b\|_{L^p(\mathbb{R})} \leqslant C \left(\frac{1}{\sqrt{n}}\right)^{r-k} \|f^{(r)} W u_B\|_{L^p(\mathbb{R})}$$

and

$$\|(L_n[f] - f)^{(k)} W u_b\|_{L^p(\mathbb{R})} \leqslant C \left(\frac{1}{\sqrt{n}}\right)^{r-k} \|f^{(r)} W (1+x^2)^{r/3} u_B\|_{L^p(\mathbb{R})},$$

where $u_\gamma(x) = (1+|x|)^\gamma$, $\gamma \in \mathbb{R}$ and $k = 0, 1, 2 \ldots, r$. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $W(x) := \mathrm{e}^{-Q(x)}$ be a Freud weight on $(-\infty, \infty)$ and $\{P_n(x)\}_{n=0}^{\infty}$ the sequence of orthonormal polynomials with respect to $W^2(x)$, that is,

$$\int_{-\infty}^{\infty} P_m(x) P_n(x) W^2(x)\, \mathrm{d}x = \delta_{mn}, \quad m, n = 0, 1, 2, \ldots \,.$$

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It is well known that all zeros of $P_n(x)$ are real and simple, which we denote by

$$x_{n,n} < x_{n-1,n} < \cdots < x_{2,n} < x_{1,n}.$$

For a given function f, let $S_n[f]$ be the nth partial sum of the orthonormal polynomials expansion

$$S_n[f](x) := \sum_{k=0}^{n-1} c_k P_k(x),$$

where

$$c_k = c_k(f) = \int_{-\infty}^{\infty} f(t)P_k(t)W^2(t)\,\mathrm{d}t, \quad k = 0, 1, \ldots, n-1.$$

We also denote by $L_n[f]$ the Lagrange interpolation polynomial of degree at most $n-1$ such that

$$L_n[f](x_{kn}) = f(x_{kn}), \quad k = 1, 2, \ldots, n.$$

Then we can write

$$L_n[f](x) = \sum_{k=1}^{n} f(x_{kn})\ell_{kn}(x),$$

where the fundamental Lagrange polynomials $\ell_{kn}(x)$ are given by

$$\ell_{kn}(x) = \frac{P_n(x)}{P_n'(x_{kn})(x - x_{kn})}, \quad k = 1, 2, \ldots, n.$$

For Freud weights, many authors have investigated the convergence of $L_n[f]$ or $S_n[f]$ to f in the weighted L^p space, see [1,2,5–7,14,15].

In this paper, we find sufficient conditions for the boundedness of $S_n[f]$ or $L_n[f]$, which implies the convergence of $S_n[f]$ or $L_n[f]$ to f in weighted L^p space, and then we obtain their speed of convergence. In case $W(x) = \mathrm{e}^{-(1/2)x^2}$ is the Hermite weight, we also give sufficient conditions for derivatives of $S_n[f]$ and $L_n[f]$ to converge to derivatives of f.

2. Preliminaries and main results

Definition 2.1. A weight function $W(x) = \mathrm{e}^{-Q(x)}$ is called a Freud weight if

(i) $Q(x)$ is even, continuous and nonnegative in $(0, \infty)$;
(ii) $Q'(x) \geqslant 0$ in $(0, \infty)$;
(iii) $Q''(x)$ is continuous in $(0, \infty)$;
(iv) there exist constants A and B such that

$$1 < A \leqslant \frac{(xQ'(x))'}{Q'(x)} \leqslant B, \quad x \in (0, \infty). \tag{2.1}$$

In investigating the mean convergence of $L_n[f]$ and $S_n[f]$, we need the Mhaskar–Rahmanov–Saff number a_n for $W(x)$, which is a positive root of the equation

$$n = \frac{2}{\pi} \int_0^1 \frac{a_n t Q'(a_n t)}{\sqrt{1 - t^2}}\,\mathrm{d}t.$$

One of the important properties for a_n is (see [12,13])

$$\|\pi W\|_{L^\infty(\mathbb{R})} = \|\pi W\|_{L^\infty[-a_n,a_n]}, \quad n \geqslant 1$$

and

$$\|\pi W\|_{L^p(\mathbb{R})} = \|\pi W\|_{L^p[-a_n,a_n]}, \quad n \geqslant 1,$$

where π is a polynomial of degree $\leqslant n$ and C is a positive constant independent of n and $\pi(x)$. The Mhaskar–Rahmanov–Saff number a_n grows roughly like the inverse of $Q(n)$ so that if $W^2(x) = e^{-|x|^\alpha}$, $\alpha > 1$, then a_n behaves like $n^{1/\alpha}$.

For any two sequences $\{c_n\}_{n=0}^\infty$ and $\{d_n\}_{n=0}^\infty$ of real numbers or functions, we use $c_n \sim d_n$ if there exist positive constants C_1 and C_2, independent of n and x, such that $C_1 d_n \leqslant c_n \leqslant C_2 d_n$, $n \geqslant 1$.

Concerning the mean convergence of $S_n[f]$, Jha and Lubinsky [5] obtained:

Theorem 2.2 (Jha and Lubinsky [5, Theorem 1.2]). *Let $1 < p < \infty$ and $W(x)$ be a Freud weight. If the inequality*

$$\|S_n[f]Wu_b\|_{L^p(\mathbb{R})} \leqslant C\|fWu_B\|_{L^p(\mathbb{R})}, \quad u_b(x) := (1+|x|)^b, \quad (b \in \mathbb{R}) \tag{2.2}$$

holds, then the following are satisfied:

(I) $b < 1 - 1/p, \quad B > -1/p, \ b \leqslant B.$

(II) *If* $p < 4/3$, *then*

$$a_n^{\max\{b,-1/p\}-B} n^{1/6(4/p-3)} L_{b,-1/p}(n) \sim O(1). \tag{2.3}$$

If $p = 4/3$ *or* $p = 4$, *then* $b < B$.

If $p > 4$, *then*

$$a_n^{b-\min\{B,1-1/p\}} n^{1/6(1-4/p)} L_{B,1-1/p}(n) \sim O(1). \tag{2.4}$$

Conversely, if conditions (I) *and* (II) *are satisfied and for* $n \geqslant 1$,

$$\sup_{x\in} |P_{n+1}(W^2;x) - P_{n-1}(W^2;x)|W(x)\left\{\left|1-\frac{|x|}{a_n}\right| + n^{-2/3}\right\}^{-1/4} \sim a_n^{-1/2}, \tag{2.5}$$

then inequality (2.2) *holds. Here, we use the notation*

$$L_{\sigma,\tau}(n) := \begin{cases} (\log n)^{|\sigma|}, & \text{if } \sigma = \tau, \\ 1 & \text{otherwise,} \end{cases}$$

It is well known that the condition (2.5) is satisfied for the typical Freud weights $W^2(x) := e^{-|x|^\alpha}$, $\alpha > 1$ (see Theorem 1.1 in [5]). But it is not easy in general to check condition (2.5) for a general Freud weight. In [7] we obtained another sufficient condition for inequality (2.2) to hold without condition (2.5). In case of $W(x) = e^{-|x|^\alpha/2}$, $\alpha > 1$, the corresponding Mhasker–Rahmanov–Saff number a_n grows like $n^{1/\alpha}$ so that conditions (2.3) and (2.4) can be simplified.

As a special case of Theorem 2.2 ($b = B = 0$), if

$$\|S_n[f]W\|_{L^p(\mathbb{R})} \leqslant C\|fW\|_{L^p(\mathbb{R})},$$

then $\frac{4}{3} < p < 4$. This is a necessary condition for the Askey–Wainger-type inequality [1] to hold for Freud weights.

Inequality (2.2) implies the convergence of $S_n[f]$ to f in weighted L^p space. In order to give a more precise result we need to state a kind of Jackson theorem by Ditzian and Lubinsky [3].

Theorem 2.3. *Let* $W(x) = \mathrm{e}^{-Q(x)}$, *where* $Q(x)$ *is even, continuous,* $Q'(x)$ *exists and is positive in* $(0, \infty)$, $xQ'(x)$ *is strictly increasing in* $(0, \infty)$ *with right limit* 0 *at* 0, *and for some* λ, $A, B > 1$, $C > 0$,

$$A \leqslant \frac{Q'(\lambda x)}{Q'(x)} \leqslant B, \quad x \geqslant C. \tag{2.6}$$

Let $r \geqslant 1$ *and* $1 \leqslant p < \infty$. *Then there exists a constant* C_1, *independent of n and f, such that*

$$E_n[f]_{W,p} := \inf_{\pi \in \mathscr{P}_n} \|(f - \pi)W\|_{L^p(\mathbb{R})} \leqslant C_1 \left(\frac{a_n}{n}\right)^r \|f^{(r)}W\|_{L^p(\mathbb{R})}$$

provided that the right-hand side is finite.

Proof. Combine Theorem 1.2 and Corollary 1.8 in [3]. □

Note that $W^2(x) = \mathrm{e}^{-|x|^\alpha}$, $\alpha > 1$, satisfies the conditions of Theorem 2.3 and relation (2.1) can be shown to imply (2.6). The restriction that $xQ'(x)$ is strictly increasing in $(0, \infty)$ may seem severe. In fact, we need it only in order to guarantee the existence of Mhaskar–Rahmanov–Saff number a_n.

Even though the modified Freud-type weight $Wu_b(x) = \mathrm{e}^{-Q(x) - b\log(1+|x|)}$ does not satisfy the conditions of Theorem 2.3, using Theorem 2.2 and some modification of Theorem 2.3 we can prove the following by the same process as in the proof of Theorem 2.3.

Theorem 2.4. *Let* $1 < p < \infty$ *and* $W(x)$ *be a Freud weight. If inequality* (2.2) *holds, then for* $f^{(r)}Wu_B \in L^p(\mathbb{R})$ *for some* $r \geqslant 1$,

$$\|(S_n[f] - f)Wu_b\|_{L^p(\mathbb{R})} \leqslant C \left(\frac{a_n}{n}\right)^r \|f^{(r)}Wu_B\|_{L^p(\mathbb{R})}.$$

In particular, if $W(x) = \mathrm{e}^{-(1/2)x^2}$ is the Hermite weight, then we can also obtain error estimates for derivatives.

Theorem 2.5. *Let* $1 < p < \infty$ *and* $W(x) = \mathrm{e}^{-(1/2)x^2}$. *If conditions* (I) *and* (II) *in Theorem* 2.2 *are satisfied, then for* $k = 0, 1, \ldots, r$,

$$\|(S_n[f] - f)^{(k)}Wu_b\|_{L^p(\mathbb{R})} \leqslant C \left(\frac{1}{\sqrt{n}}\right)^{r-k} \|f^{(r)}Wu_B\|_{L^p(\mathbb{R})}.$$

Proof. Let $W^2(x) = \mathrm{e}^{-x^2}$ and $H_n(x)$ be the Hermite polynomials. Since

$$H_n''(x) - 2xH_n'(x) = -2nH_n(x), \quad n \geqslant 0$$

and $H_n'(x) = \sqrt{2n}H_{n-1}(x)$, we have by integration by parts,

$$c_j(f') = \int_{-\infty}^{\infty} f'(t)H_j(t)\mathrm{e}^{-t^2}\,\mathrm{d}t = \sqrt{2(j+1)}c_{j+1}(f).$$

Hence,

$$S_n[f'](x) = \sum_{k=0}^{n-1} c_k(f')H_k(x) = \sum_{k=0}^{n-1} \sqrt{2(k+1)}c_{k+1}(f)H_k(x)$$
$$= \sum_{k=0}^{n-1} c_{k+1}(f)H'_{k+1}(x) = \sum_{k=0}^{n} c_k(f)H'_k(x) = S'_{n+1}[f](x).$$

Inductively, we obtain $S_{n+k}^{(k)}[f](x) = S_n[f^{(k)}](x)$. Thus, by Theorem 2.4, we have for $k = 0, 1, \ldots, r$,

$$\|(S_n^{(k)}[f] - f^{(k)})Wu_b\|_{L^p(\mathbb{R})} = \|(S_{n-k}[f^{(k)}] - f^{(k)})Wu_b\|_{L^p(\mathbb{R})}$$
$$\leqslant C\left(\frac{a_{n-k}}{n-k}\right)^{r-k} \|f^{(r)}Wu_B\|_{L^p(\mathbb{R})}.$$

Since $a_n = \sqrt{2n}$ and $a_n/n \sim a_{n-k}/(n-k)$, the conclusion follows. □

Concerning the boundedness of $L_n[f]$, we first introduce the following [6] which are generalizations of results in [11].

Lemma 2.6 (Jung et al. [6]). *Let $W(x)$ be a Freud weight and r a positive integer. Then for $g^{(r-1)} \in AC_{\mathrm{loc}}$ and $g^{(r)}(1+Q)^{r/3}W^2 \in L^1(\mathbb{R})$,*

$$R_n(W^2; g) := \left|\int_{-\infty}^{\infty} (g(x) - L_n[g](x))W^2(x)\,\mathrm{d}x\right| \leqslant C_1\left(\frac{a_n}{n}\right)^r \|g^{(r)}(1+Q)^{r/3}W^2\|_{L^1(\mathbb{R})}. \tag{2.7}$$

Moreover, if $g^{(r-1)} \in AC_{\mathrm{loc}}$ and $g^{(r)}W^2 \in L^1(\mathbb{R})$, then

$$R_n(W^2; g) = \left|\int_{-\infty}^{\infty} (g(x) - L_n[g](x))W^2(x)\,\mathrm{d}x\right| \leqslant C_2\left(\frac{a_n}{n^{2/3}}\right)^r \|g^{(r)}W^2\|_{L^1(\mathbb{R})}. \tag{2.8}$$

Theorem 2.7. *Let $1 < p < \infty$ and $W(x)$ be a Freud weight. Assume that b and B satisfy the conditions* (I), (II), *and* (2.5) *in Theorem* 2.2 *and assume moreover that the Markov–Bernstein type inequality*

$$\|P'Wu_{-B}\|_{L^q(\mathbb{R})} \leqslant C_1\left(\frac{n}{a_n}\right)\|PWu_{-B}\|_{L^q(\mathbb{R})}, \quad P \in \mathscr{P} \quad \left(q = \frac{p}{p-1}\right) \tag{2.9}$$

holds. Then for $f^{(r-1)} \in AC_{\mathrm{loc}}$ and $f^{(r)}(1+Q)^{r/3}Wu_B \in L^p(\mathbb{R})$,

$$\|L_n[f]Wu_b\|_{L^p(\mathbb{R})} \leqslant C_2\left(\|fWu_B\|_{L^p(\mathbb{R})} + \sum_{j=0}^{r}\binom{r}{j}\left(\frac{a_n}{n}\right)^j \|f^{(j)}(1+Q)^{r/3}Wu_B\|_{L^p(\mathbb{R})}\right). \tag{2.10}$$

Moreover, if $f^{(r-1)} \in AC_{\mathrm{loc}}$ and $f^{(r)}Wu_B \in L^p(\mathbb{R})$, then

$$\|L_n[f]Wu_b\|_{L^p(\mathbb{R})} \leqslant C_3\left(\|fWu_B\|_{L^p(\mathbb{R})} + \sum_{j=0}^{r}\binom{r}{j}\left(\frac{a_n}{n^{2/3}}\right)^j \|f^{(j)}Wu_B\|_{L^p(\mathbb{R})}\right). \tag{2.11}$$

Here we use the notation $u_\gamma(x) = (1+|x|)^\gamma$, $\gamma \in \mathbb{R}$.

For general Markov–Bernstein-type inequalities we refer to [4,8,9,17] and references therein. Let $W_\gamma(x) := e^{-Q_\gamma(x)}$, where $Q_\gamma(x) = Q(x) - \gamma \log(1 + |x|)$. If $W(x)$ is a regular Freud weight (see [8, Definition 1.2]), then by Theorem 2.4 in [8], $W_\gamma(x)$ is also a regular Freud weight. Moreover, it is easy to see that $q_n \sim a_n(Wu_{-B})$, where q_n is the Freud number in Corollary 3.4 in [8] so that the Markov–Bernstein-type inequality (2.9) holds for modified Freud weight Wu_{-B}, $B \in \mathbb{R}$. We refer to [8] for various examples of regular Freud weights. For example,

$$W(x) = (1 + x^2)^\gamma \exp(-|x|^\alpha (\log(1 + |x|))^{\beta_1} (\log_2(1 + |x|))^{\beta_2} \cdots (\log_k(1 + |x|))^{\beta_k})$$

is a regular Freud weight, where $\gamma \in \mathbb{R}$, $\alpha > 0$ and $\beta_j \geqslant 0$, $j = 1, 2, \ldots, k$, while $\log_k$ denotes the kth iterated logarithm.

We also recall that the typical Freud weight $W^2(x) = e^{-|x|^\alpha}$, $\alpha > 1$, satisfies conditions (I), (II), and (2.5) in Theorem 2.2. Hence, the conditions of Theorem 2.7 are satisfied at least for the weights $W^2(x) = e^{-|x|^\alpha}$, $\alpha > 1$.

Proof. The proof is very similar to that given in [2]. First, note that the Lagrange interpolation polynomial can be written as

$$L_n[f](x) = \sum_{j=1}^{n} \lambda_{jn} f(x_{jn}) K_n(x, x_{jn}),$$

where $K_n(x, t) = \sum_{j=0}^{n-1} P_j(x)P_j(t)$ is the reproducing kernel and λ_{jn} ($j = 1, 2, \ldots, n$) are the Christoffel numbers with respect to the weight $W^2(x)$. Let $G(x) = \operatorname{sgn} L_n[f](x)$ and $F(x) = |L_n[f](x)|$. Then

$$\begin{aligned}
\|L_n[f]Wu_b\|^p_{L^p(\mathbb{R})} &= \int_{-\infty}^{\infty} L_n[f](x) F^{p-1}(x) G(x) W^p(x) u_b^p(x)\, dx \\
&= \sum_{j=1}^{n} \lambda_{jn} f(x_{jn}) \int_{-\infty}^{\infty} K_n(x, x_{jn}) F^{p-1}(x) G(x) W^p(x) u_b^p(x)\, dx \\
&= \sum_{j=1}^{n} \lambda_{jn} f(x_{jn}) \pi_{n-1}(x_{jn}),
\end{aligned} \tag{2.12}$$

where

$$\begin{aligned}
\pi_{n-1}(t) &= \int_{-\infty}^{\infty} K_n(x, t) F^{p-1}(x) G(x) W^p(x) u_b^p(x)\, dx \\
&= S_n[F^{p-1} G W^{p-2} u_b^p](t).
\end{aligned}$$

By (2.7) and (2.12), we have

$$\begin{aligned}
\|L_n[f]Wu_b\|^p_{L^p(\mathbb{R})} &\leqslant \int_{-\infty}^{\infty} |f(x)\pi_{n-1}(x)| W^2(x)\, dx + |R_n(W^2, f\pi_{n-1})| \\
&\leqslant \|fWu_B\|_{L^p(\mathbb{R})} \|\pi_{n-1} Wu_{-B}\|_{L^q(\mathbb{R})} \\
&\quad + C_4 \left(\frac{a_n}{n}\right)^r \|(f\pi_{n-1})^{(r)} (1 + Q)^{r/3} W^2\|_{L^1(\mathbb{R})}
\end{aligned}$$

$$
\begin{aligned}
&= \|fWu_B\|_{L^p(\mathbb{R})}\|\pi_{n-1}Wu_{-B}\|_{L^q(\mathbb{R})}\\
&\quad + C_4\left(\frac{a_n}{n}\right)^r \sum_{k=0}^{r}\binom{r}{k}\|f^{(k)}\pi_{n-1}^{(r-k)}(1+Q)^{r/3}W^2\|_{L^1(\mathbb{R})}\\
&\leqslant \|fWu_B\|_{L^p(\mathbb{R})}\|\pi_{n-1}Wu_{-B}\|_{L^q(\mathbb{R})}\\
&\quad + C_4\left(\frac{a_n}{n}\right)^r \sum_{k=0}^{r}\binom{r}{k}\|f^{(k)}(1+Q)^{r/3}Wu_B\|_{L^p(\mathbb{R})}\|\pi_{n-1}^{(r-k)}Wu_{-B}\|_{L^q(\mathbb{R})}.
\end{aligned}
$$

By the Markov–Bernstein inequality (2.9), we have

$$\|\pi_{n-1}^{(r-k)}Wu_{-B}\|_{L^q(\mathbb{R})} \leqslant C_5\left(\frac{n}{a_n}\right)^{r-k}\|\pi_{n-1}Wu_{-B}\|_{L^q(\mathbb{R})}$$

and so

$$
\begin{aligned}
&\|L_n[f]Wu_b\|_{L^p(\mathbb{R})}^p\\
&\quad\leqslant C_6\|\pi_{n-1}Wu_{-B}\|_{L^q(\mathbb{R})}\left(\|fWu_B\|_{L^p(\mathbb{R})} + \sum_{j=0}^{r}\binom{r}{j}\left(\frac{a_n}{n}\right)^j\|f^{(j)}(1+Q)^{r/3}Wu_B\|_{L^p(\mathbb{R})}\right). \qquad (2.13)
\end{aligned}
$$

Since conditions (I), (II) and (2.5) are satisfied, by Theorem 2.2 and duality, conditions (I) and (II) also hold for $-B$ and $-b$ if p is replaced by $q = p//(p-1)$ so that

$$\|S_n[f]Wu_{-B}\|_{L^q(\mathbb{R})} \leqslant C_7\|fWu_{-b}\|_{L^q(\mathbb{R})}.$$

Hence,

$$
\begin{aligned}
\|\pi_{n-1}Wu_{-B}\|_{L^q(\mathbb{R})} &= \|S_n[F^{p-1}GW^{p-2}u_b^p]Wu_{-B}\|_{L^q(\mathbb{R})}\\
&\leqslant C_8\|F^{p-1}GW^{p-1}u_b^{p-1}\|_{L^q(\mathbb{R})}\\
&= C_8\|L_n[f]Wu_b\|_{L^p(\mathbb{R})}^{p-1}. \qquad (2.14)
\end{aligned}
$$

Substituting (2.14) into (2.13), we obtain inequality (2.10). Inequality (2.11) can be obtained by the same process using (2.8) instead of (2.7). □

Since a_n grows like a polynomial as $n \to \infty$, if the Markov–Bernstein-type inequality holds for Wu_b, then we can inductively deduce from Theorem 2.7 that for $k = 0, 1, \ldots, r$,

$$\|L_n^{(k)}[f]Wu_b\|_{L^p(\mathbb{R})} \leqslant C_9\left(\frac{n}{a_n}\right)^k\left(\|fWu_B\|_{L^p(\mathbb{R})} + \sum_{j=0}^{r}\binom{r}{j}\left(\frac{a_n}{n}\right)^j\|f^{(j)}(1+Q)^{r/3}Wu_B\|_{L^p(\mathbb{R})}\right) \qquad (2.15)$$

and

$$\|L_n^{(k)}[f]Wu_b\|_{L^p(\mathbb{R})} \leqslant C_{10}\left(\frac{n}{a_n}\right)^k\left(\|fWu_B\|_{L^p(\mathbb{R})} + \sum_{j=0}^{r}\binom{r}{j}\left(\frac{a_n}{n^{2/3}}\right)^j\|f^{(j)}Wu_B\|_{L^p(\mathbb{R})}\right). \qquad (2.16)$$

On the mean convergence of Lagrange interpolation polynomial $L_n[f]$ to f, there are many results when $W(x)$ is a Freud weight. For example we refer to [10,16]. But as far as the authors know, there is no result treating the convergence of the derivatives of $L_n[f]$ even in the case of $W(x)=e^{-(1/2)x^2}$. Note that inequalities (2.15) and (2.16) do not directly imply the convergence of $L_n^{(k)}[f]$ to $f^{(k)}$ in weighted L^p space. But here we can get some conditions under which $L_n^{(k)}[f]$ converges to $f^{(k)}$ as well as showing the speed of its convergence when $W(x)$ is the Hermite weight function.

Theorem 2.8. *Let $b, B \in$, $1 < p < \infty$, $W(x)=e^{-(1/2)x^2}$, and r be a positive integer. Assume that*

$$b < 1 - \frac{1}{p}, \quad B + \frac{2r}{3} > -\frac{1}{p}, \quad B + \frac{2r}{3} \geqslant b \tag{2.17}$$

and

- *if $1 < p < 4/3$, then*
$$\max\left\{b, -\frac{1}{p}\right\} - \min\left\{B + \frac{2r}{3}, 1 - \frac{1}{p}\right\} + \frac{2}{3}\left(\frac{4}{p} - 3\right) < 0: \tag{2.18}$$
- *if $p > 4$, then*
$$\max\left\{b, -\frac{1}{p}\right\} - \min\left\{B + \frac{2r}{3}, 1 - \frac{1}{p}\right\} + \frac{2}{3}\left(1 - \frac{4}{p}\right) < 0. \tag{2.19}$$

Then for $f^{(r-1)} \in AC_{\text{loc}}$ and $f^{(r)}(1+x^2)^{r/3}Wu_B \in L^p(\mathbb{R})$, and $k = 0, 1, \ldots, r$,

$$\|(f - L_n[f])^{(k)}Wu_b\|_{L^p(\mathbb{R})} \leqslant C\left(\frac{1}{\sqrt{n}}\right)^{r-k} \|f^{(r)}(1+x^2)^{r/3}Wu_B\|_{L^p(\mathbb{R})}, \tag{2.20}$$

where C is independent of n and f.

Proof. Since $W(x)=e^{-(1/2)x^2}$, $(1+Q(x))^{(1/3)r} \sim (1+|x|)^{(2/3)r}$ so that

$$\|f(1+Q)^{r/3}Wu_B\|_{L^p(\mathbb{R})} \sim \|fWu_{B+2r/3}\|_{L^p(\mathbb{R})}.$$

Let us introduce a parameter $\tilde{b}$ such that

$$\tilde{b} = -\frac{2}{3}r + \frac{1}{2}\left(\max\left\{b, -\frac{1}{p}\right\} + \min\left\{B + \frac{2}{3}r, 1 - \frac{1}{p}\right\}\right). \tag{2.21}$$

Then a simple calculation with (2.17), (2.18), (2.19) and (2.21) shows that

(III) $b < 1 - 1/p$, $\tilde{b} + 2r/3 > -1/p$, $b \leqslant \tilde{b} + 2r/3$;
(IV) If $p < 4/3$, then

$$\max\{b, -1/p\} - \tilde{b} - 2r/3 + \frac{1}{3}(4/p - 3)\begin{cases} \leqslant 0 & \text{if } b \neq -1/p, \\ < 0 & \text{if } b = -1/p. \end{cases}$$

If $p = 4/3$ or $p = 4$, then $b < \tilde{b} + 2r/3$.
If $p > 4$, then

$$b - \min\{\tilde{b} + 2r/3, 1 - 1/p\} + \frac{1}{3}(1 - 4/p)\begin{cases} \leqslant 0 & \text{if } \tilde{b} \neq 1 - 1/p - 2r/3, \\ < 0 & \text{if } \tilde{b} = 1 - 1/p - 2r/3. \end{cases}$$

Since $a_n \sim \sqrt{2n}$, conditions (I) and (II) in Theorem 2.2 are satisfied for the couple b and $\tilde{b}+2r/3$. Similarly, conditions (I) and (II) in Theorem 2.2 are also satisfied for the couple $\tilde{b}+2r/3$ and $B+2r/3$. Hence by Theorem 2.2, we have

$$\|S_n[f]W_b\|_{L^p(\mathbb{R})} \leqslant C_{18}\|fW_{\tilde{b}+2r/3}\|_{L^p(\mathbb{R})}; \quad \|S_n[f]W_{\tilde{b}+2r/3}\|_{L^p(\mathbb{R})} \leqslant C_{19}\|fW_{B+2r/3}\|_{L^p(\mathbb{R})}$$

and so

$$\begin{aligned}
&\|(f-L_n[f])W_b\|_{L^p(\mathbb{R})}\\
&\quad\leqslant \|(f-S_n[f])W_b\|_{L^p(\mathbb{R})} + \|L_n[f-S_n[f]]W_b\|_{L^p(\mathbb{R})}\\
&\quad\leqslant C_{20}\left(\frac{1}{\sqrt{n}}\right)^r \|f^{(r)}W_{\tilde{b}+2r/3}\|_{L^p(\mathbb{R})}\\
&\qquad + C_{21}\left(\|(f-S_n[f])W_{\tilde{b}+2r/3}\|_{L^p(\mathbb{R})} + \sum_{j=0}^{r}\binom{r}{j}\left(\frac{1}{\sqrt{n}}\right)^j \|(f-S_n[f])^{(j)}W_{\tilde{b}+2r/3}\|_{L^p(\mathbb{R})}\right)\\
&\quad\leqslant C_{22}\left(\frac{1}{\sqrt{n}}\right)^r\left(\|f^{(r)}W_{B+2r/3}\|_{L^p(\mathbb{R})} + C_{23}\sum_{j=0}^{r}\binom{r}{j}\|f^{(r)}W_{B+2r/3}\|_{L^p(\mathbb{R})}\right)\\
&\quad\leqslant C_{24}\left(\frac{1}{\sqrt{n}}\right)^r \|f^{(r)}W_{B+2r/3}\|_{L^p(\mathbb{R})}.
\end{aligned}$$

For the Hermite weight $W(x)=\mathrm{e}^{-(1/2)x^2}$, conditions (III) and (IV) imply conditions (I) and (II) in Theorem 2.2 and so Theorem 2.5 is applicable. Using Theorem 2.5, we can prove (2.20) by a process similar to that above for $k=0,1,\ldots,r$. □

Theorem 2.8 gives the speed of convergence for the derivatives of Lagrange interpolation polynomial in case of $W(x)=\mathrm{e}^{-(1/2)x^2}$. In particular, if $\frac{4}{3}<p<4$, then for $k=0,1,\ldots,r$,

$$\|(f-L_n[f])^{(k)}W\|_{L^p(\mathbb{R})} \leqslant C\left(\frac{1}{\sqrt{n}}\right)^{r-k}\|f^{(r)}W(1+x^2)^{r/3}\|_{L^p(\mathbb{R})}.$$

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